A series of coverings of the regular n-gon

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Abstract

We define an infinite series of translation coverings of Veech's double-n-gon for odd $n \geq 5$ which share the same Veech group. Additionally we give an infinite series of translation coverings with constant Veech group of a regular n-gon for even $n \geq 8$. These families give rise to explicit examples of infinite translation surfaces with lattice Veech group.

1 Introduction

Billiards in a polygon with angles commensurable with π, can be investigated by considering the geodesic flow on an appropriate translation surface (see [ZK75]). The fundamental work of Veech in 1989 connects the properties of the geodesic flow to the so called *Veech group* of the *translation surface* (see [Vee89]). The *Veech alternative* states, that if the Veech group is a lattice, then the billiard flow in each direction is either periodic or uniquely ergodic.

Some translation surfaces which have the lattice property already considered by Veech himself, are the double-n-gons X_n for odd $n \geq 5$. They arise from billiards in a triangle with angles π/n, π/n and $(n-2)\pi/n$. For even n, $n \geq 8$, the regular double-n-gons considered by Veech, are degree 2 coverings of the regular n-gons X_n. The Veech groups of all these surfaces are well known lattice groups.

For every odd $n \geq 5$ and every $d \geq 2$ we define a *translation covering* $p_{n,d} : Y_{n,d} \to X_n$ of degree d and calculate the Veech group of $Y_{n,d}$.

Theorem 1 a). *For odd $n \geq 5$, the Veech group of $Y_{n,d}$ is*

$$\Gamma_n := \Gamma(Y_{n,d}) = \langle -I, T_n, R_n\, {T_n}^2\, {R_n}^{-1}, \dots, {R_n}^{\frac{n-1}{2}}\, {T_n}^2\, {R_n}^{-\frac{n-1}{2}} \rangle\ ,$$

where

$$R_n = \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix}\ , \quad T_n = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix}\ , \quad \lambda_n = 2\cot\frac{\pi}{n}$$

and $I \in \mathrm{GL}_2(\mathbb{R})$ is the identity matrix.

If we likewise define translation coverings $p_{n,d} : Y_{n,d} \to X_n$ in the even case for $n \geq 8$ and $d \geq 2$, we get a similar statement.

Theorem 1 b). *For even $n \geq 8$, the Veech group of $Y_{n,d}$ is*

$$\begin{aligned} \Gamma_n := \ & \Gamma(Y_{n,d}) \\ = \ & \langle -I, T_n, {R_n}^2\, {T_n}^2\, {R_n}^{-2}, \dots, {R_n}^{n-2}\, {T_n}^2\, {R_n}^{-(n-2)}, ({T_n}^{-1}{R_n}^2)^2, \\ & {R_n}^2\, ({T_n}^{-1}{R_n}^2)^2\, {R_n}^{-2}, \dots, {R_n}^{n-2}\, ({T_n}^{-1}{R_n}^2)^2\, {R_n}^{-(n-2)} \rangle \ , \end{aligned}$$

where R_n, T_n, λ_n and I are defined as in Theorem 1 a).

So for all $n \geq 5$, $n \neq 6$ we obtain an infinite family of translation coverings with fixed Veech group Γ_n. The *Teichmüller curve* arising from the translation surface $Y_{n,d}$ has the following properties.

Corollary. *$\mathbb{H}/\Gamma_n$ has genus 0, $\frac{n+1}{2}$ cusps if n is odd and $\frac{n+2}{2}$ cusps if n is even.*

The limit of each family is an infinite translation surface, $Y_{n,\infty}$.

Theorem 2. *For $n \geq 5$, $n \neq 6$, the Veech group $\Gamma(Y_{n,\infty})$ of $Y_{n,\infty}$ is Γ_n. In particular $Y_{n,\infty}$ is an infinite translation surface with a lattice Veech group.*

So the families give rise to explicit examples of infinite translation surfaces with lattice Veech groups, which are rare by now. One example of an infinite translation surface with lattice Veech group was found in [HW09]. It is a $\mathbb{Z}$-cover of a double cover of the regular octagon. A second example is the translation surface described in [Hoo08]. It is build from two infinite polygons, the convex hull of the points (n, n^2) and the convex hull of the points $(n, -n^2)$. Another example is the infinite staircase origami $Z_{2,0}^{\infty}$, calculated in [HS10].

We will treat the case where n is odd in detail. The proof for even n works very similar, hence we keep it short.

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2 Basic definitions and preliminaries.

In this section we want to shortly review the basic definitions used to state and prove our theorems. For a more detailed introduction to Veech groups see e.g. [HS01] or [Vor96]. More details on Teichmüller curves can be found e.g. in [HS07].

A *(finite) translation surface* X is a compact Riemann surface with a finite, nonempty set $\Sigma(X)$ of *singular points* together with a maximal 2-dimensional atlas ω on $X \setminus \Sigma(X)$, such that all transition functions between the charts are translations. The 2-dimensional atlas ω induces a flat metric on $X \setminus \Sigma(X)$, whereas the angles around points in $\Sigma(X)$ are integral multiples of 2π. If the

angle around a singular point is 2π, then the flat metric can be extended to that point, otherwise the flat metric has a conical singularity. A translation surface is obtained by gluing finitely many polygons via identification of edge pairs using translations. In this construction, singularities may arise from the vertices of the polygons. Especially in this situation the translation structure on X is somehow obvious, so we often omit ω in the notation.

A continuous map $p : Y \to X$ is called a *translation covering*, if $p(\Sigma(Y)) = \Sigma(X)$ and $p : Y \setminus \Sigma(Y) \to X \setminus \Sigma(X)$ is locally a translation. Since both X and Y are compact, p is a finite covering map in the topological sense, ramified at most over the singularities $\Sigma(X)$. In the context of translation coverings $p : Y \to X$, we call X the *base surface* and Y the *covering surface* of p. A translation surface (X, ω) that, in this sense, is not the covering surface of a base surface with smaller genus, is called *primitive*. In the following, only base surfaces of genus greater than one are considered and all singularities have conical angle greater than 2π, so $p^{-1}(\Sigma(X)) = \Sigma(Y)$. This kind of translation coverings are often called *balanced* translation coverings. It is a result of Möller in [MÖ06], that every translation surface is the (balanced) covering surface of a primitive base surface and that the primitive surface is unique, if its genus is greater than one.

The *affine group* of a translation surface (X, ω) is the set of affine, orientation preserving diffeomorphisms on X, i.e. maps that can locally be written as $z \mapsto Az + b$ with $A \in \mathrm{SL}_2(\mathbb{R})$ and $b \in \mathbb{R}^2$. The translation vector b depends on the local coordinates, whereas the derivative A is globally defined. The derivatives of the affine maps on X (i.e. of the elements in the affine group), form the *Veech group* $\Gamma(X)$ of X. An important connection between the Veech groups of a primitive base surface and the covering surface in a translation covering is stated in the following lemma.

Lemma 2.1. *Let $p : (Y, \nu) \to (X, \omega)$ be a (finite, balanced) translation covering with primitive base surface (X, ω) and genus $g(X) > 1$, then $\Gamma(Y)$ is a subgroup of $\Gamma(X)$.*

The Lemma is an immediate consequence of the result of Möller cited above (see [McM06]). In Chapter 5, the statement of Lemma 2.1 will be proved separately as Lemma 5.3 for finite and infinite coverings of the base surfaces X_n, which are introduced in Chapter 3.

We will often use the *cylinder decomposition* of a translation surface X. A *cylinder* is a maximal connected set of homotopic simple closed geodesics. The *inverse modulus* of a cylinder is the ratio of its length to its height. Whenever the genus of X is greater than one, every cylinder is bounded by geodesic intervals joining singular points. Such a geodesic is called *saddle connection* if no singular point lies in the interior. The *Veech alternative* states, that if the Veech group of a translation surface is a lattice, then the geodesic flow in each direction is either periodic or uniquely ergodic. This important result of Veech implies in particular, that if X has a lattice Veech group and if there exists a closed geodesic in direction θ on X then the surface decomposes into cylinders in direction θ.

Being a compact Riemann surface, X defines a point in the moduli space $M_{g,s}$ of Riemann surfaces of genus g with $s = |\Sigma(X)|$ punctures. Every matrix $A \in \mathrm{SL}_2(\mathbb{R})$ can be used to change the translation structure ω on X, by composing each chart with the affine map $z \mapsto A \cdot z$. We call the new translation surface X_A. The identity map $\mathrm{id}_A : X \to X_A$ is an orientation preserving homeomorphism, so (X_A, id_A) is a point in the Teichmüller space $T_{g,s}$. The surfaces X_A and X_B define the same point in $T_{g,s}$ iff $A \cdot B^{-1} \in \mathrm{SO}_2(\mathbb{R})$ thus defining a map $i : \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \cong \mathbb{H} \hookrightarrow T_{g,s}$. The Veech group $\Gamma(X)$ lies in the stabiliser in $\mathrm{SL}_2(\mathbb{R})$ of X_I as Riemann surface. Hence we obtain a map from $\mathbb{H}/\Gamma(X)$ to $M_{g,s}$. By the Theorem of Smillie the image of this map, which is birational to $\mathbb{H}/\Gamma(X)$, is an algebraic curve in the moduli space iff $\Gamma(X)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. In this case, this curve is called a *Teichmüller curve.*

3 The base surfaces.

3.1 The odd regular double-n-gon.

The translation surface described in this section, the odd regular double-n-gon X_n, was already considered by Veech himself in [Vee89]. Other references concerning this translation surface are [HS01] Chapter 1.7 and [Vor96] Chapter 4. The surface arises from a triangle with angles π/n, π/n and $(n-2)\pi/n$ by the unfolding construction described in [ZK75].

In the following, let n be an odd number, greater or equal to 5. The translation surface X_n can be glued of two regular n-gons P_n and Q_n. Rotate the n-gons until each of them has a horizontal side and P_n lies above its horizontal side, whereas Q_n lies below it. Gluing parallel sides leads to the desired translation surface. To fix lengths, let the circumscribed circle of P_n (respectively of Q_n) have radius 1. Then the edges of the n-gons have lengths $2 \cdot \sin(\frac{\pi}{n})$. We

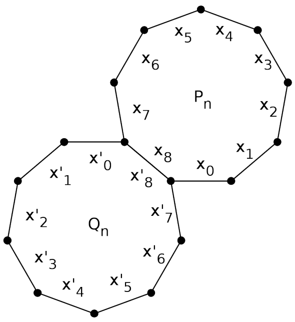


Figure 1: Labelling of X_9

label the edges of the two polygons in the following way: Label the edges of P_n and Q_n counter-clockwise with $x_0, x_1, \ldots, x_{n-1}$ and $x'_0, x'_1, \ldots, x'_{n-1}$, starting with the horizontal side (see Figure 1). Whenever needed, we initially glue the edges x_{n-1} and x'_{n-1} to obtain only one polygon that defines X_n.

Remark 3.1. Following the identification of vertices while gluing the edges x_i and x'_i one can easily see that all the vertices are identified. X_n thereby has

exactly one singularity with conical angle $2n\,\frac{n-2}{n}\pi = (n-2)\,2\pi$. The Euler characteristic is $\chi = 1 - n + 2 = 3 - n$ and the genus of X_n is $g(X_n) = \frac{n-1}{2}$.

According to [Vee89] Theorem 5.8, the Veech group of X_n is generated by the matrices

$$R_n := \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix} \quad \text{and} \quad T_n := \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix} \quad \text{where} \quad \lambda_n = 2\cot\frac{\pi}{n}\,.$$

Let φ_R denote the orientation preserving affine diffeomorphism of X_n with derivative R_n. The map φ_R rotates X_n around the centre m_P of P_n and further translates X_n along the vector from m_P to the centre m_Q of Q_n.

To understand the affine map on X_n with derivative T_n, we use a cylinder decomposition of X_n. The horizontal saddle connections in X_n decompose the translation surface into $\frac{n-1}{2}$ cylinders. Figure 2 indicates how to compute the heights and lengths of these cylinders: First we rotate the double-n-gon counterclockwise by 90 degrees. If we choose the origin of a coordinate system in the centre of P_n, then the vertices of P_n lie in $(\cos(j\pi\,\frac{2}{n}), \sin(j\pi\,\frac{2}{n}))$, $j = 0, \dots, n-1$. This is all the information we need.

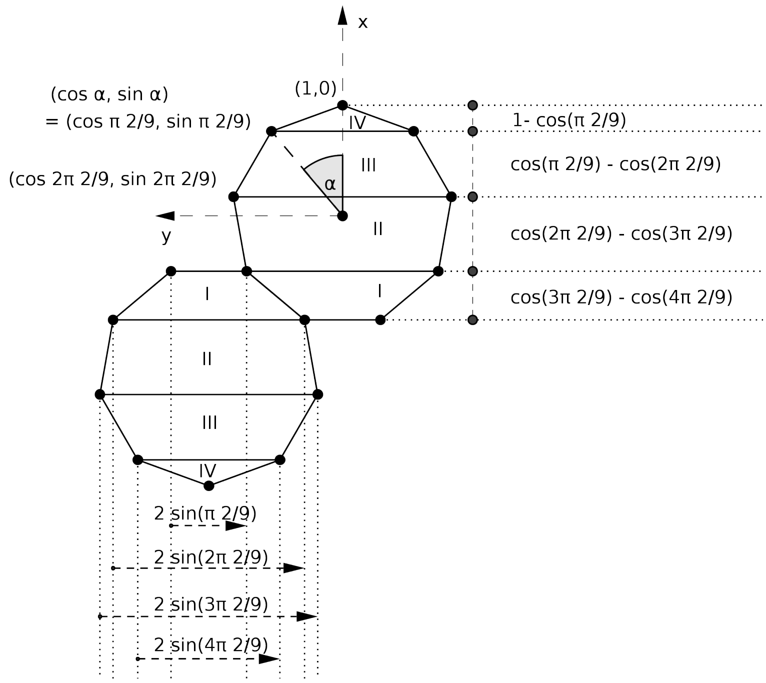


Figure 2: Horizontal cylinder decomposition of X_9

We name the cylinder containing the gluing along the edges x_i and x_{n-i} with i (see Figure 2). If h_i denotes the height and l_i the length of the ith cylinder and $j = \frac{n+1}{2} - i$, then

$$\begin{aligned} h_i &= \cos\tfrac{2\pi}{n}(j-1) - \cos\tfrac{2\pi}{n}j \\ &= -2\sin\tfrac{\pi(2j-1)}{n}\sin\tfrac{-\pi}{n} = 2\sin\tfrac{\pi(2j-1)}{n}\sin\tfrac{\pi}{n} \end{aligned} \tag{1}$$

and

$$\begin{aligned} l_i &= 2(\sin \tfrac{2\pi}{n} j + \sin \tfrac{2\pi}{n}(j-1)) \\ &= 4 \sin \tfrac{\pi(2j-1)}{n} \cos \tfrac{\pi}{n} \end{aligned} \tag{2}$$

for $i \in \{1, \ldots, \frac{n-1}{2}\}$. Note that we used the following consequence of the addition and subtraction theorems for sin and cos:

$$\begin{aligned} \cos x - \cos y &= -2\sin(\tfrac{x+y}{2})\sin(\tfrac{x-y}{2}) \\ \sin x + \sin y &= 2\sin(\tfrac{x+y}{2})\cos(\tfrac{x-y}{2}) \end{aligned}$$

The inverse modulus in all horizontal cylinders of X_n is $\lambda_n = 2\cot\frac{\pi}{n}$. Thus we can construct an orientation preserving affine diffeomorphism φ_T of X_n with derivative T_n (see e.g. [Vor96] Chapter 3.2). The map φ_T fixes all horizontal saddle connections pointwise and twists every cylinder once.

Remark 3.2. The Veech group $\Gamma(X_n)$ is the Hecke triangle group with signature $(2, n, \infty)$, thus it is a lattice.

It is well known, that the translation surface X_n is primitive. We give a short proof nevertheless.

Lemma 3.3. *The translation surface X_n with odd $n \geq 5$ is primitive.*

Proof. Suppose $p : X_n \to Y$ is a translation covering of degree $d > 1$. The surface X_n has one singularity, say x, so p is ramified at most over $p(x)$. We have to distinguish two cases.

If $y = p(x)$ is a removable singularity in the flat structure, than Y is w.l.o.g. the once punctured torus. According to [GJ00] a translation surface is the covering surface of the once punctured torus if and only if its Veech group is arithmetic, i.e. commensurable to $\mathrm{SL}_2(\mathbb{Z})$. But the Veech group of $\Gamma(X_n)$ is not arithmetic.

In the second case, y is a non removable singularity in the flat structure. Then $p^{-1}(y) = \{x\}$. It follows that the preimage of a saddle connection on Y is the union of d saddle connections on X_n. The map p is locally a translation, so theses d saddle connections all have the same length and direction. Since there is only one shortest saddle connection on X_n in horizontal direction, this leads to a contradiction to $d > 1$.

We conclude that every translation covering $p : X_n \to Y$ has degree 1, i.e. the genus of Y equals the genus of X. $\square$

In particular X_n has no translations, with the consequence that there is exactly one affine map with derivative A for each A in the Veech group $\Gamma(X_n)$.

According to Remark 3.1, the translation surface X_n has one singularity and genus $\frac{n-1}{2}$. Consequently the fundamental group $\pi_1(X_n \setminus \Sigma)$ is free of rank $n-1$. We use the centre of the side x_{n-1} as base point. A basis of the fundamental group of the double-n-gon can be chosen in such a way, that the ith generator ($i \in \{0, \ldots, n-2\}$) crosses the edge x_i once from P_n to Q_n (and respectively the edge x_{n-1} once from Q_n to P_n). The ith generator of the fundamental group will be called x_i, like the label of the edge it crosses (see Figure 3). Now an arbitrary element of the fundamental group $\pi_1(X_n \setminus \Sigma)$ can be factorised in this basis by recording the names of the crossed edges and the directions of the crossings. The edge x_{n-1} corresponds to the identity element.

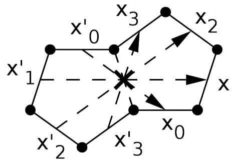
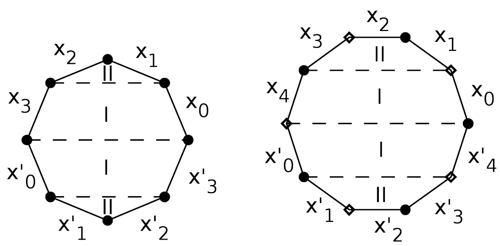


Figure 3: Fundamental group of the double-5-gon

3.2 The even regular n-gon.

Now let n be an even number and $n \geq 8$. The translation surface X_n arises from a regular n-gon P, with opposite sides glued together. As indicated in the introduction, the regular double n-gon, considered by Veech in [Vee89], is a degree-2-covering of this surface. It is well known, that the regular n-gon is a primitive translation surface. To fix the size and orientation of the polygon, let the jth vertex of P lie in $(\cos(j\frac{2\pi}{n}), \sin(j\frac{2\pi}{n}))$, $j \in \{0, \ldots, n-1\}$. We label the edges P counter-clockwise with $x_0, \ldots, x_{n/2-1}, x'_0, \ldots, x'_{n/2-1}$, starting with the edge between vertex 0 and vertex 1 (see Figure 4).



(a) X_8 (b) X_{10}

Figure 4: Labelling of X_n for even n

Remark 3.4. The surface X_n has one singularity, if $n/2$ is even and two singularities if $n/2$ is odd. Using the Euler characteristic it follows, that X_n has genus

$$g(X_n) = \begin{cases} n/4 & \text{, if } n \equiv 0 \bmod 4 \\ (n-2)/4 & \text{, if } n \equiv 2 \bmod 4 \end{cases} .$$

Lemma J in [HS01] states, that the Veech group of X_n equals the Veech group of Veech's double-n-gons for even $n \geq 8$. So according to [Vee89] Theorem 5.8, $\Gamma(X_n) = \langle T_n, {R_n}^2, R_n T_n {R_n}^{-1} \rangle$ where

$$R_n := \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix} \quad , \quad T_n := \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \lambda_n = 2\cot\frac{\pi}{n}$$

are defined as in the odd case. Since $R_n T_n {R_n}^{-1} = {R_n}^{n+2} {T_n}^{-1}$ (see relations on page 11), only the first two generators are needed.

Again we need the cylinder decomposition of X_n in horizontal direction to understand the affine map on X_n with derivative T_n. The surface X_n decomposes into $n/4$ or $(n-2)/4$ cylinders, depending on $n \equiv 0 \bmod 4$ or $n \equiv 2 \bmod 4$.

If we label the cylinder containing the gluing along the edges i and $(n/2-1)-i$ with $i+1$ (see Figure 4), we get

$$h_i = 2\cos\frac{(2i-1)\pi}{n}\sin\frac{\pi}{n} \quad \text{and} \quad l_i = 4\cos\frac{(2i-1)\pi}{n}\cos\frac{\pi}{n} \tag{3}$$

for $i \in \{1,\ldots,\frac{n}{4}\}$ or $i \in \{1,\ldots,\frac{n-2}{4}\}$, respectively. It follows that the inverse modulus is $2\cot\frac{\pi}{n} = \lambda_n$ in all horizontal cylinders.

Remark 3.5. For even n, the rotation R_n is not contained in the Veech group $\Gamma(X_n)$. In this case $\Gamma(X_n)$ is a triangle group with signature $(\frac{n}{2},\infty,\infty)$ and an index 2 subgroup of the Hecke triangle group $\langle T_n, R_n\rangle$ with signature $(2,n,\infty)$.

The translation surface X_n has one singularity and genus $n/4$ if $n \equiv 0 \bmod 4$ or two singularities and genus $(n-2)/4$ if $n \equiv 2 \bmod 4$. In both cases, the fundamental group $\pi_1(X_n \setminus \Sigma)$ is free of rank $n/2$. We use the centre as base point and choose closed paths that cross exactly one edge x_i one time as basis of the fundamental group (see Figure 5). Now an arbitrary element of the fundamental group $\pi_1(X_n \setminus \Sigma)$ can be factorised in this basis by recording the names of the crossed edges and the directions of the crossings.

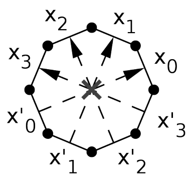


Figure 5: Fundamental group of the regular 8-gon

4 A series of n-gon coverings.

Using a generalised algorithm of the one presented in [Sch04], it is possible to calculate the Veech group of an arbitrary finite covering of a double-n-gon or n-gon (see [Fre08]). With the help of such computations, a conjecture about an interesting family of translation coverings

$$p_{n,d} : Y_{n,d} \to X_n$$

arose. In the following we define the family $p_{n,d}$ and prove that the Veech group of the covering surface $Y_{n,d}$ is

$$\Gamma(Y_{n,d}) = \langle -I, T_n, R_n\,{T_n}^2\,{R_n}^{-1}, \ldots, {R_n}^{\frac{n-1}{2}}\;{T_n}^2\,{R_n}^{-\frac{n-1}{2}}\rangle$$

for odd $n \geq 5$ and

$$\begin{aligned}\Gamma(Y_{n,d}) = \quad & \langle -I, T_n, {R_n}^2\,{T_n}^2\,{R_n}^{-2}, \ldots, {R_n}^{n-2}\,{T_n}^2\,{R_n}^{-(n-2)},\ ({T_n}^{-1}{R_n}^2)^2,\\ & {R_n}^2\,({T_n}^{-1}{R_n}^2)^2\,{R_n}^{-2}, \ldots, {R_n}^{n-2}\,({T_n}^{-1}{R_n}^2)^2\,{R_n}^{-(n-2)}\rangle\end{aligned}$$

for even $n \geq 8$ with R_n and T_n as in Chapter 3.1. In particular, it is independent of the covering degree d. Our proof uses geometric arguments and not the methods, used to develop the algorithm mentioned above.

The series of coverings is somehow similar to the stair-origamis in [Sch06] and to the $Z^k_{2,0}$ series in [HS10]. It likewise uses two slits in the base surface to construct the covering surface and has a Veech group that is independent of the covering degree.

We will define the coverings by their monodromy. We recall the definition of the monodromy; for more details see [Mir95] Chapter III.4. Let $p_{n,d} : Y_{n,d} \to X_n$ be a covering of degree d and $\Sigma = \Sigma(X_n)$. Choose a base point x in $X_n \setminus \Sigma$ and call its preimages in $Y_{n,d}$ $0, \dots, d-1$. Every closed path in $X_n \setminus \Sigma$ at x can be lifted to a path in $Y_{n,d}$ with starting point in $\{0, \dots, d-1\}$. The end point of the lifted path is again contained in $\{0, \dots, d-1\}$ and the lifts define a permutation of the points $\{0, \dots, d-1\}$ in $Y_{n,d}$. The corresponding map

$$m : \pi_1(X_n \setminus \Sigma, x) \to S_d$$

has the following property

$$m([w_1] \cdot [w_2]) = m([w_2]) \circ m([w_1]) \quad \forall\, [w_1], [w_2] \in \pi_1(X_n \setminus \Sigma, x)$$

and is called the *monodromy* of the covering. Its image in S_d is a transitive permutation group. On the other hand, every such map $m : \pi_1(X_n \setminus \Sigma, x) \to S_d$ with transitive image defines a degree d covering of X_n.

To define the map m it is sufficient to define the images of the generators $x_0, \dots, x_{n-2}$ (or $x_0, \dots, x_{\frac{n}{2}-1}$ respectively) of $\pi_1(X_n \setminus \Sigma, x)$ in S_d. The generator x_i crosses the edge x_i and no other edge, so that the permutation $\sigma_i = m(x_i)$ indicates directly how the d copies of X_n are glued along the edges x_i and x_i' to obtain the covering surface $Y_{n,d}$. The edge x_i in copy j is glued to the edge x_i' in copy $\sigma_i(j)$.

4.1 Definition of the coverings.

In the following, we define for each $d \geq 2$ a covering $p : Y_{n,d} \to X_n$ by giving its monodromy $m_{n,d}$. For the definition, we need the two permutations

$$\sigma_{d,1} = \begin{cases} (0\ 1)\,(2\ 3)\ \cdots\ (d-2\ \ d-1) & , d \text{ even} \\ (0\ 1)\,(2\ 3)\ \cdots\ (d-3\ \ d-2) & , d \text{ odd} \end{cases}$$

and

$$\sigma_{d,2} = \begin{cases} (1\ 2)\,(3\ 4)\ \cdots\ (d-3\ \ d-2)\,(d-1\ \ 0) & , d \text{ even} \\ (1\ 2)\,(3\ 4)\ \cdots\ (d-2\ \ d-1) & , d \text{ odd} \end{cases}.$$

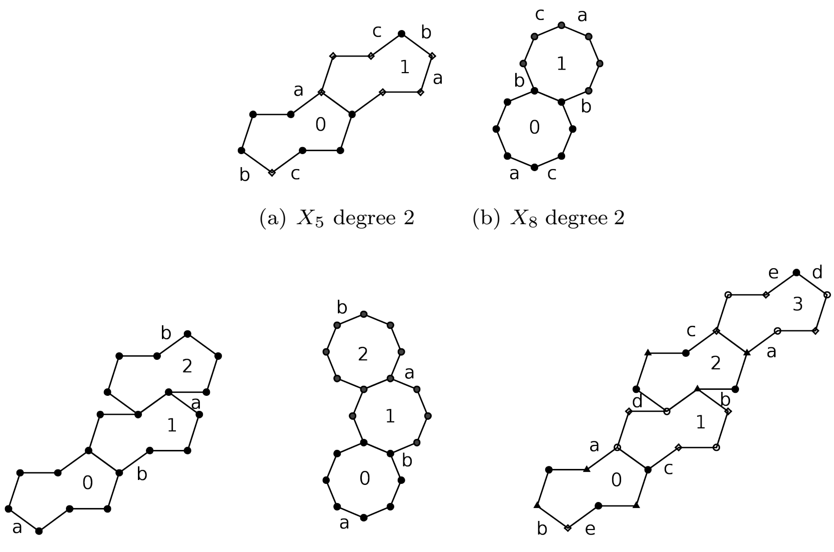
Further, let

$$k_1 = \begin{cases} \frac{n-1}{2} & , n \text{ odd} \\ \frac{n}{4} - 1 & , n \equiv 0 \bmod 4 \\ \frac{n-2}{4} - 1 & , n \equiv 2 \bmod 4 \end{cases} \quad \text{and} \quad k_2 = \begin{cases} \frac{n+1}{2} & , n \text{ odd} \\ \frac{n}{4} & , n \equiv 0 \bmod 4 \\ \frac{n-2}{4} + 1 & , n \equiv 2 \bmod 4 \end{cases}.$$

For $d \geq 2$ define the monodromy $m_{n,d}$ by:

$$m_{n,d} : \begin{cases} \pi_1(X_n \setminus \Sigma) & \longrightarrow & S_d & \\ x_i & \mapsto & \text{id} & , i \notin \{k_1, k_2\} \\ x_{k_1} & \mapsto & \sigma_{d,1} & \\ x_{k_2} & \mapsto & \sigma_{d,2} & \end{cases}$$

Let $Y_{n,d}$ denote the covering surface of the translation covering $p : Y_{n,d} \to X_n$, defined by the monodromy $m_{n,d} : F_{n-1} \to S_d$ or $m_{n,d} : F_{\frac{n}{2}} \to S_d$, respectively. Figure 6 shows the degree 2, 3 and 4 coverings of the double-5-gon, as well as the degree 2 and 3 covering of the 8-gon. The non labelled edges are glued to the parallel edge in the same X_n copy, labelled edges are glued to their labelled correspondent.



(a) X_5 degree 2 (b) X_8 degree 2

(c) X_5 degree 3 (d) X_8 degree 3 (e) X_5 degree 4

Figure 6: Translation coverings $Y_{n,d}$ of X_n

Figure 7 and Figure 8 show the action of the generators of $\pi_1(X_n \setminus \Sigma)$ via $m_{n,d}$ on the set $\{0, \ldots, d-1\}$.

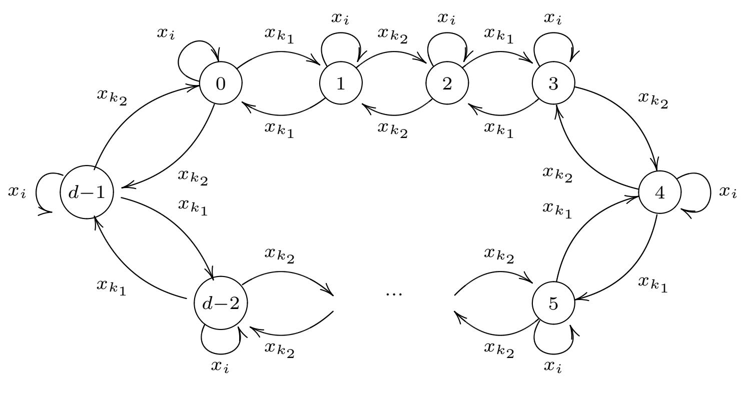


Figure 7: Monodromy action for even d

Remark 4.1. We excluded $n = 4$ and $n = 6$ in our considerations, because the genus of X_4 and X_6 is one. In these cases the singularities of the base surface X_n are removable (i.e. have angle 2π). Hence it is an additional assumption, that

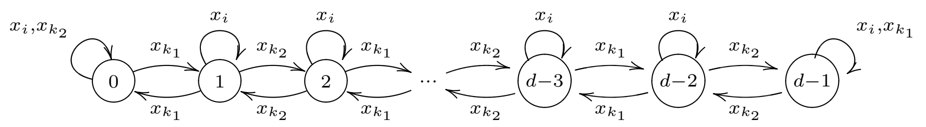


Figure 8: Monodromy action for odd d

affine maps on the surface map singularities to singularities and that covering maps $p : Y \to X_n$ satisfy $p^{-1}(\Sigma(X_n)) = \Sigma(Y)$. If $n = 4$ the covering surface $Y_{4,d}$ for even degree $d = 2l$ is the lattice surface $\Gamma_{2,0}^{l}$ defined in [HS10] and the limit of our series is their lattice surface $\Gamma_{2,0}^{\infty}$. For $n = 6$, considerations similar to Proposition 2.6. in [Sch04] show that every affine map f on Y (respecting $\Sigma(Y) = p^{-1}(\Sigma(X_6))$) descends via a translation covering $p : Y \to X_6$ to an affine map $\tilde{f}$ on the twice punctured torus X_6 with $p \circ f = \tilde{f} \circ p$, i.e. $\Gamma(Y) \subseteq \Gamma(X_6) = \langle T_6, {R_6}^2 \rangle$. By considering some extra cases in the proof, it can be seen that Theorem 1 b) also holds for $n = 6$.

4.2 The Veech group of the odd series.

The goal of this section is the proof of our main theorem for odd $n \geq 5$.

Theorem 1 a). *For odd $n \geq 5$, the Veech group of $Y_{n,d}$ is*

$$\Gamma_n := \Gamma(Y_{n,d}) = \langle -I, T_n, R_n\, {T_n}^2\, {R_n}^{-1}, \ldots, {R_n}^{\frac{n-1}{2}}\ {T_n}^2\, {R_n}^{-\frac{n-1}{2}} \rangle\ .$$

The matrices R_n and T_n are defined as in Chapter 3.1 as

$$R_n = \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix} \text{ and } T_n = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix} \text{ where } \lambda_n = 2\cot\frac{\pi}{n}$$

and $I \in \mathrm{GL}_2(\mathbb{R})$ is the identity matrix.

In particular, Γ_n is a subgroup of $\Gamma(X_n)$ of index n (see Lemma 4.2) and independent of the covering degree d. At the end of this section, we deduce the basic properties of the Teichmüller curve defined by the translation surface $Y_{n,d}$ in Corollary 4.3. We start the proof of the theorem with the following lemma.

Lemma 4.2. *The group $G := \langle -I, T_n, R_n\, {T_n}^2\, {R_n}^{-1}, \ldots, {R_n}^{\frac{n-1}{2}}\ {T_n}^2\, {R_n}^{-\frac{n-1}{2}} \rangle$ is of index n in $\Gamma(X_n) = \langle R_n, T_n \rangle$. The set of cosets is*

$$G\backslash\Gamma(X_n) = \{G\, I, G\, R_n, \ldots, G\, {R_n}^{n-1}\}\,,$$

where $I \in \mathrm{GL}_2(\mathbb{R})$ denotes the identity matrix.

Proof. Write $\Gamma(X_n) = F/N$, where F is the free group on the generators R and T and N is the normal subgroup corresponding to the relations of R and T in $\Gamma(X_n)$. The image of $\Gamma(X_n)$ in $\mathrm{PSL}_2(\mathbb{R})$ is a triangle group with signature $(\infty, n, 2)$, so a possible set of defining relations in $\mathrm{PSL}_2(\mathbb{R})$ is $\{R^n, (T^{-1}\, R)^2\}$. An easy calculation shows, that $R^n = -I = (T^{-1}\, R)^2$. We deduce (see e.g. [Joh97] Chapter 10), that a possible set of defining relations of $\Gamma(X_n)$ in $\mathrm{SL}_2(\mathbb{R})$ is $\{(T^{-1}\, R)^2 = R^n, R^{2n} = I, R^nT = TR^n\}$.

The method of Schreier (see e.g. [Joh97] Chapter 2) shows, that

$$SchreierGen = \{R^n, T, R\,T\,(R^{n-1})^{-1}, \ldots, R^{n-1}\,T\,R^{-1}\}$$

is a free basis of an index n subgroup U of F with Schreier transversal

$$\mathcal{T} = \{I, R, \ldots, R^{n-1}\}\,.$$

Let $p : F \to \Gamma(X_n)$ denote the projection from F onto $\Gamma(X_n)$. Our first claim is that $p(U) = G$.

To simplify notation we skip the index n in the following computations, i.e. $p(R) = R$ and $p(T) = T$. The inclusion $G \subset p(U)$ is immediate: The first two generators of G belong to $p(SchreierGen)$, the other generators are contained in $\langle p(SchreierGen)\rangle$ since $R^jT^2R^{-j} = R^jT(R^{n-j})^{-1} \cdot R^{n-j}TR^{-j}$. The inclusion $p(U) \subset G$ needs some more arithmetic. The relation $R^n = -I = (T^{-1}\,R)^2$ implies $TR^{-1}T = -R$ and $T^{-1} = -R^{-1}TR^{-1}$. It follows, that $R^jT(R^{n-j})^{-1} \in G$ for $1 \leq j \leq \frac{n-1}{2}$:

$$\begin{aligned}
& (R^n)^j \cdot R^jT^2R^{-j} \cdot R^{j-1}T^2R^{-(j-1)} \cdot \ldots \cdot R^2T^2R^{-2} \cdot RT^2R^{-1} \cdot T \cdot (R^n)^{-1} \\
= \; & \underbrace{(R^n)^j}_{(-I)^j} \cdot R^jT\,\underbrace{TR^{-1}T}_{-R}\,\underbrace{TR^{-1}\ldots}_{-R}\ldots\underbrace{\ldots R^{-1}T}_{-R}\,\underbrace{TR^{-1}T}_{-R}\,\underbrace{TR^{-1}\cdot T}_{-R} \cdot R^{-n} \\
= \; & (-I)^j(-I)^jR^jTR^{j-n} = R^jT(R^{n-j})^{-1}
\end{aligned}$$

If $\frac{n-1}{2} < j < n-1$ then $0 < n-1-j < \frac{n-1}{2}$, so that $R^{n-1-j}T(R^{n-(n-1-j)})^{-1} = R^{n-1-j}TR^{-j-1} \in G$. Because of

$$R^n \cdot (R^{n-1-j}TR^{-j-1})^{-1} = -I \cdot R^{j+1} \underbrace{T^{-1}}_{-R^{-1}TR^{-1}} R^{-n+j+1} = R^jTR^{-n+j}$$

this implies $R^jT(R^{n-j})^{-1} \in G$ for $\frac{n+1}{2} \leq j \leq n-2$. The equation $R^n \cdot R^n \cdot T^{-1} = -I \cdot R^n \cdot (-R^{-1}TR^{-1}) = R^{n-1}TR^{-1}$ completes the proof of the claim.

It remains to show, that the kernel of p is contained in U. Then the index $[\Gamma(X_n) : G] = [F : U] = n$ with $\{I, R, \ldots, R^{n-1}\}$ a system of coset representatives. The kernel of the map p is the normal closure of the set $Rel = \{R^n(T^{-1}\,R)^{-2}, R^{2n}, R^nTR^{-n}T^{-1}\}$, so that we need to show $Uwr = Uw$ $\forall\, w \in F, r \in Rel$. Because $\{I, R, \ldots, R^{n-1}\}$ is a system of coset representatives for U, this is equivalent to $R^jrR^{-j} \in U \;\forall\, r \in Rel,\, j \in \{1, \ldots, n-1\}$.

$$\begin{aligned}
R^j \cdot R^n(T^{-1}\,R)^{-2} \cdot R^{-j} &= R^{n+j-1}TR^{-1}TR^{-j} \\
&= R^n \cdot R^{j-1}T(R^{n-(j-1)})^{-1} \cdot R^{n-j}TR^{-j} \in U \\
R^j \cdot R^{2n} \cdot R^{-j} &= R^{2n} = (R^n)^2 \in U \\
R^j \cdot R^nTR^{-n}T^{-1} \cdot R^{-j} &= R^n \cdot R^jT(R^{n-j})^{-1}R^{-n}R^{n-j}T^{-1}R^{-j} \\
&= R^n \cdot R^jT(R^{n-j})^{-1} \cdot R^{-n} \cdot (R^jTR^{-n+j})^{-1} \in U
\end{aligned}$$

□

The surface X_n is primitive and has genus greater than one, so $\Gamma(Y_{n,d})$ is a subgroup of $\Gamma(X_n)$ (see Lemma 2.1). In combination with Lemma 4.2 the proof of Theorem 1 a) reduces to the proof of

(i) $\{-I, T_n, R_n\, T_n{}^2\, R_n^{-1}, \ldots, R_n{}^{\frac{n-1}{2}}\; T_n{}^2\, R_n{}^{-\frac{n-1}{2}}\} \subset \Gamma_n = \Gamma(Y_{n,d})$ and

(ii) $I, R_n, \ldots, R_n{}^{n-1}$ lie in different cosets of $\Gamma_n \backslash \Gamma(X_n)$.

The rest of the proof of Theorem 1 a) is divided into four steps. The first step is to prove, that the parabolic matrices $R_n\, T_n{}^2\, R_n{}^{-1}, \ldots, R_n{}^{\frac{n-1}{2}}\; T_n{}^2\, R_n{}^{-\frac{n-1}{2}}$ with shearing factor $2\lambda_n$ are contained in Γ_n. Afterwards we show, that I, $R_n, \ldots, R_n{}^{n-1}$ lie in different cosets of $\Gamma_n \backslash \Gamma(X_n)$. The last two steps will show that $-I$ and T_n belong to Γ_n.

Step 1: $\boldsymbol{R_n{}^l\, T_n{}^2\, R_n{}^{-l} \in \Gamma_n}$ for $\boldsymbol{1 \leq l \leq \frac{n-1}{2} = k_1}$

The affine map on $\mathbb{R}^2$, $x \mapsto M_l \cdot x$ with

$$M_l = R_n{}^l\, T_n{}^2\, R_n{}^{-l} = \begin{pmatrix} 1 - 2\lambda_n \cos(\frac{l\pi}{n}) \sin(\frac{l\pi}{n}) & 2\lambda_n \cos^2(\frac{l\pi}{n}) \\ -2\lambda_n \sin^2(\frac{l\pi}{n}) & 1 + 2\lambda_n \cos(\frac{l\pi}{n}) \sin(\frac{l\pi}{n}) \end{pmatrix}$$

is the shearing in direction $v_l = R_n{}^l \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(l\pi/n) \\ \sin(l\pi/n) \end{pmatrix}$ with factor $2\lambda_n$. To find an affine map on $Y_{n,d}$ with derivative M_l, we investigate the cylinder decomposition of $Y_{n,d}$ in direction v_l.

Because of the rotational symmetry in X_n the cylinder decomposition of X_n in direction v_l has the same properties as the horizontal cylinder decomposition (see page 5). The vector v_l lies in the same direction as the edge $x_{l/2}$ if l is even and in the same direction as the edge $x_{k_1+(l+1)/2}$, if l is odd. The cylinders in the cylinder decomposition in direction x_i contain gluings along the following edges (where all the indices are understood to be modulo n):

- cylinder 1: x_{i-1} and x_{i+1}
- cylinder 2: x_{i-2} and x_{i+2}

  $\vdots$

- cylinder $\frac{n-1}{2}$: $x_{i-\frac{n-1}{2}}$ and $x_{i+\frac{n-1}{2}}$

Since the monodromy map for $Y_{n,d}$ sends x_i to the identity in S_d iff $i \notin \{k_1, k_2\}$, the d copies of X_n are connected only by the edges x_{k_1} and x_{k_2}. Copies of cylinders in X_n that contain neither x_{k_1} nor x_{k_2} are glued to themselves in $Y_{n,d}$. They keep their inverse modulus λ_n. Cylinders that contain either x_{k_1} or x_{k_2} are glued according to $m_{n,d}(x_{k_1})$ or to $m_{n,d}(x_{k_2})$ respectively. For every cycle of length a in $m_{n,d}(x_{k_1})$ (or in $m_{n,d}(x_{k_2})$), a copies of the cylinder containing x_{k_1} (or x_{k_2}) are glued to form one cylinder in $Y_{n,d}$ (see Figure 9). Since both $m_{n,d}(x_{k_1})$ and $m_{n,d}(x_{k_2})$ have only cycles of length 2, the inverse modulus of theses cylinders is $2\lambda_n$. Non of the vectors v_l is horizontal ($l \neq 0$), so that no cylinder decomposition in direction v_l has a cylinder containing both edges x_{k_1} and x_{k_2}.

We may conclude, that M_l is the derivative of the affine map ϕ_l on $Y_{n,d}$, fixing each saddle connection in direction v_l pointwise and twisting the cylinders in the cylinder decomposition in direction v_l once or twice (again see e.g. [Vor96] Chapter 3.2 for a more detailed explanation of such an affine map). It follows that $R_n{}^l\, T_n{}^2\, R_n{}^{-l} \in \Gamma_n$ for $1 \leq l \leq k_1$.

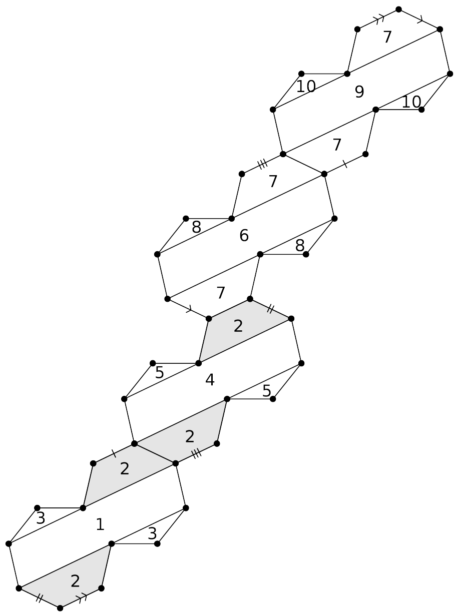


Figure 9: Diagonal cylinder decomposition

Step 2: $I, R_n, \dots, {R_n}^{n-1}$ lie in different cosets of $\Gamma_n \backslash \Gamma(X_n)$

It is sufficient to show, that ${R_n}^l \notin \Gamma_n \; \forall \, l \in \{1, \dots, n-1\}$, since

$$\Gamma(X_n) \cdot {R_n}^i = \Gamma(X_n) \cdot {R_n}^j \Leftrightarrow {R_n}^{i-j} \in \Gamma(X_n) \,.$$

In this part of the proof we use the horizontal cylinder decomposition of X_n. In this decomposition, cylinder $k_1 = \frac{n-1}{2}$ contains the gluings along the edges x_{k_1} and x_{k_2}. More precisely, a closed horizontal path in the cylinder k_1 describes an element in $\langle x_{k_1} {x_{k_2}}^{-1} \rangle \subset \pi_1(X_n \setminus \Sigma)$. As above we conclude that for every cycle of length a in $m_{n,d}(x_{k_1} {x_{k_2}}^{-1})$ a copies of the cylinder k_1 are glued in $Y_{n,d}$ to form one cylinder.

If d is even, then:

$$\begin{aligned}
& m_{n,d}(x_{k_1} {x_{k_2}}^{-1}) = m_{n,d}({x_{k_2}}^{-1}) \circ m_{n,d}(x_{k_1}) \\
& = (\,(1\,2)\,(3\,4)\,\cdots\,(d-3\;\;d-2)\,(d-1\;\;0)\,)^{-1} \circ (0\,1)\,(2\,3)\,\cdots\,(d-2\;\;d-1) \\
& = (1\,2)\,(3\,4)\,\cdots\,(d-3\;\;d-2)\,(d-1\;\;0) \circ (0\,1)\,(2\,3)\,\cdots\,(d-2\;\;d-1) \\
& = (0\;\;2\;\;4 \dots d-2)\,(1\;\;d-1\;\;d-3 \dots 3)
\end{aligned}$$

If d is odd, it follows:

$$\begin{aligned}
& m_{n,d}(x_{k_1} {x_{k_2}}^{-1}) = m_{n,d}({x_{k_2}}^{-1}) \circ m_{n,d}(x_{k_1}) \\
& = (1\,2)\,(3\,4)\,\cdots\,(d-2\;\;d-1) \circ (0\,1)\,(2\,3)\,\cdots\,(d-3\;\;d-2) \\
& = (0\;\;2\;\;4 \dots d-3\;\;d-1\;\;d-2\;\;d-4 \dots 3\;\;1)
\end{aligned}$$

Therefore the inverse modulus of the horizontal cylinders of $Y_{n,d}$, which map onto the cylinder k_1 of X_n, is $\frac{d}{2}\lambda_n$ if d is even and $d\lambda_n$ if d is odd. All the other horizontal cylinders in $Y_{n,d}$ have inverse modulus λ_n.

The conclusions made in step 1 about the cylinders in the direction v_l do not rely on l to be smaller or equal to $\frac{n-1}{2}$. They also apply for $1 \leq l \leq n-1$. We conclude that all the cylinders in direction v_l have inverse modulus λ_n or $2\lambda_n$. Now suppose ϕ_{R^l} to be an affine map on $Y_{n,d}$ with derivative ${R_n}^l$. Then ϕ_{R^l} would send the horizontal cylinders onto the cylinders in direction v_l. As a rotation is length preserving, the moduli of the cylinders in horizontal direction would have to match the moduli of the ones in direction v_l. It is an immediate consequence for $d \notin \{2, 4\}$, that ${R_n}^l$ is not contained in Γ_n. Simply, because there is no cylinder in direction v_l with inverse modulus $\frac{d}{2}\lambda_n$ or $d\lambda_n$ respectively.

For $d = 2$, all the cylinders in horizontal direction have inverse modulus λ_n. In all other directions v_l, there is no cylinder containing both edges x_{k_1} and x_{k_2}. On the other hand, only one of the edges x_{k_1} or x_{k_2} can lie in direction v_l, so that there is at least one cylinder containing one of x_{k_1} or x_{k_2}. This cylinder has only one preimage cylinder $\tilde{c}$ in $Y_{n,d}$, so that the cylinder $\tilde{c}$ has inverse modulus $2\lambda_n$, which leads to a contradiction.

If $d = 4$, then there are 2 cylinders with inverse modulus $2\lambda_n$ in the horizontal decomposition of $Y_{n,d}$. For all directions v_l, where x_{k_1} and x_{k_2} do not lie in direction v_l, $Y_{n,d}$ has 4 cylinders with inverse modulus $2\lambda_n$, 2 from x_{k_1} and 2 from x_{k_2}. In the remaining cases, were $d = 4$ and v_l is parallel to x_{k_1} or x_{k_2}, the moduli of the cylinders in the decomposition in direction v_l match the ones of the horizontal decomposition. Here we have to examine the heights of the cylinders with inverse modulus $2\lambda_n$. The cylinder $k_1 = \frac{n-1}{2}$ and of course also its preimages, have the height $2\sin(\frac{\pi}{n})\sin(\frac{\pi}{n})$ (see equation 1 on page 5). The cylinder of X_n in direction v_l which is parallel to x_{k_1} or x_{k_2} and contains the gluing along x_{k_2} or x_{k_1} respectively, is the cylinder 1 (i.e. the image of cylinder 1 in the horizontal decomposition under ${R_n}^l$). It has the height $2\sin((n-2)\frac{\pi}{n})\sin(\frac{\pi}{n})$. Using the consequence of the addition and subtraction theorems for sin and cos

$$\sin x - \sin y = 2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2}) \ ,$$

the difference of the heights of the cylinders with inverse modulus $2\lambda_n$ in $Y_{n,d}$ can be simplified as follows:

$$\begin{aligned}
&2\sin(\tfrac{\pi}{n})\sin(\tfrac{\pi}{n}) - 2\sin(\tfrac{\pi}{n})\sin((n-2)\tfrac{\pi}{n}) \\
&= 2\sin(\tfrac{\pi}{n})\cos(\tfrac{n-1}{2}\cdot\tfrac{\pi}{n})\sin(\tfrac{-n+3}{2}\cdot\tfrac{\pi}{n}) \quad \neq 0
\end{aligned}$$

Since a rotation can not map cylinders with different heights onto each other, this case also leads to a contradiction.

Step 3: $T_n \in \Gamma_n$

To show, that T_n is contained in the Veech group of X_n, we use again the horizontal cylinder decomposition of X_n. In the following we construct an affine map ϕ_T on $Y_{n,d}$ with derivative T_n.

As shown above, for all but the outermost cylinder of X_n the preimage on $Y_{n,d}$ decomposes into d cylinders with inverse modulus λ_n, respectively. An affine map ϕ_T with derivative T_n on $Y_{n,d}$ twists these cylinders once and possibly

permutes them, with say σ_T. Of course, if an inner cylinder j in copy i is mapped onto cylinder j in copy $\sigma_T(i)$, then all the other inner cylinders in copy i are mapped onto their correspondent in copy $\sigma_T(i)$. The preimages of cylinder $\frac{n-1}{2}$ in X_n (now called c) form 1 or 2 cylinders in $Y_{n,d}$, depending on d being odd or even. These cylinders are sheared by the factor λ_n.

Recall, that $Y_{n,d}$ is a degree d translation covering of X_n, in particular $Y_{n,d}$ contains d copies of every point in X_n, except for the vertices which may be ramification points. The preimage of c in one copy of X_n is not connected, so from now on we use a slightly different labelling. To get the new labelling we cut the lower part of c in each copy along the horizontal saddle connection and glue it along the edge x_{k_1}, i.e. the lower part of c in copy i now "belongs" to the copy $m_{n,d}({x_{k_1}}^{-1})(i)$ (see Figure 10).

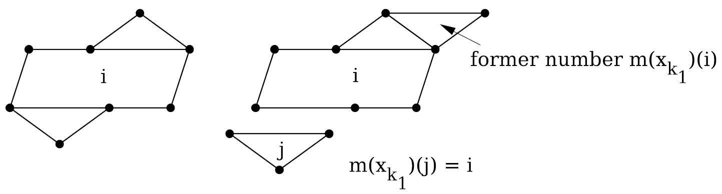


Figure 10: New labelling of the outermost cylinders

Every new copy i of c has as lower neighbour copy i of $X_n \setminus c$ and as upper neighbour copy $m_{n,d}(x_{k_1})(i)$ of $X_n \setminus c$ (see Figure 11).

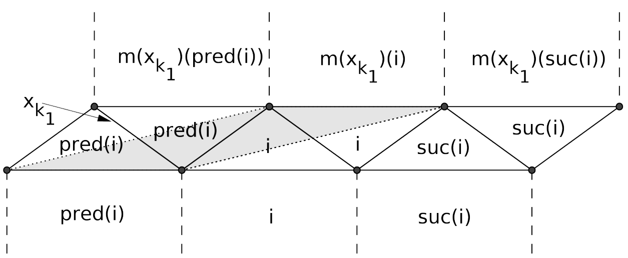


Figure 11: Schematic drawing of the outermost cylinder

The preimage cylinder(s) of c define a natural order of the d copies:

$$\text{suc}(i) = m_{n,d}(x_{k_1}{x_{k_2}}^{-1})(i) \qquad \text{pred}(i) = m_{n,d}(x_{k_2}{x_{k_1}}^{-1})(i)$$

All we need to prove is the existence of a permutation σ_T of the copies of $X_{n,d}$ that fulfils the following two conditions:

1. The lower neighbours retain their order, i.e. the image of the successor of i equals the successor of the image of i:

$$\sigma_T(\text{suc}(i)) = \text{suc}(\sigma_T(i))$$

2. The upper neighbours are shifted by one, i.e. the upper neighbour of the image of i is the image of the upper neighbour of the predecessor of i:

$$m_{n,d}(x_{k_1})(\sigma_T(i)) = \sigma_T(m_{n,d}(x_{k_1})(\text{pred}(i)))$$

Using the definition of pred and $m_{n,d}(a \cdot b) = m_{n,d}(b) \circ m_{n,d}(a)$, the equation can be written as

$$m_{n,d}(x_{k_1})(\sigma_T(i)) = \sigma_T(m_{n,d}(x_{k_2})(i))\,.$$

Such a permutation induces an affine map ϕ_T as follows:
The inner cylinders of copy i are mapped to the inner cylinders of copy $\sigma_T(i)$ and further twisted once. The preimages of c are sheared by the factor λ_n. The lower bounding saddle connection of copy i of c is mapped onto the corresponding saddle connection in copy $\sigma_T(i)$, the upper saddle connection to its correspondent in copy $\sigma_T(\mathrm{suc}(i))$ (in the new labelling). The two conditions above ensure, that ϕ_T respects the gluings.

Claim. For even d, $\sigma_T = (1\ \ 3\ \ 5 \ldots d-3\ \ d-1)$ satisfies condition 1 and 2.

Proof.

$$\begin{aligned}
\sigma_T \circ \mathrm{suc} &= (1\ \ 3\ \ 5 \ldots d-3\ \ d-1) \circ (0\ \ 2\ \ 4 \ldots d-2)\,(1\ \ d-1\ \ d-3 \ldots 3)\\
&= (0\ \ 2\ \ 4 \ldots d-2)\\
&= (0\ \ 2\ \ 4 \ldots d-2)\,(1\ \ d-1\ \ d-3 \ldots 3) \circ (1\ \ 3\ \ 5 \ldots d-3\ \ d-1)\\
&= \mathrm{suc} \circ \sigma_T
\end{aligned}$$

$$\begin{aligned}
m_{n,d}(x_{k_1}) \circ \sigma_T &= (0\,1)\,(2\,3)\,\cdots\,(d-2\ \ d-1) \circ (1\ \ 3\ \ 5 \ldots d-3\ \ d-1)\\
&= (0\ \ 1\ \ 2\ \ 3 \ldots d-2\ \ d-1)\\
&= (1\ \ 3 \ldots d-3\ \ d-1) \circ (0\ \ d-1)\,(1\,2)\,\cdots\,(d-3\ \ d-2)\\
&= \sigma_T \circ m_{n,d}(x_{k_2})
\end{aligned}$$

□

Claim. For odd d, $\sigma_T = m((x_{k_1}x_{k_2})^{\frac{d-1}{2}})$ satisfies condition 1 and 2.

Proof. Recall, that $m_{n,d}(x_{k_1}) = \sigma_{d,1}$, $m_{n,d}(x_{k_2}) = \sigma_{d,2}$ and that $\sigma_{d,1}$ as well as $\sigma_{d,2}$ is of order 2. Since the permutation $m_{n,d}(x_{k_1}x_{k_2}) = \sigma_{d,2} \circ \sigma_{d,1} = (0\ \ 2\ \ 4 \ldots d-3\ \ d-1\ \ d-2\ \ d-4 \ldots 3\ \ 1)$ is of order d, it follows that $(\sigma_{d,2} \circ \sigma_{d,1})^d = \mathrm{id}$ and the second condition

$$\sigma_{d,1} \circ (\sigma_{d,2} \circ \sigma_{d,1})^{\frac{d-1}{2}} = (\sigma_{d,2} \circ \sigma_{d,1})^d \circ \sigma_{d,1} \circ (\sigma_{d,2} \circ \sigma_{d,1})^{\frac{d-1}{2}} = (\sigma_{d,2} \circ \sigma_{d,1})^{\frac{d-1}{2}} \circ \sigma_{d,2}$$

holds. The first condition is immediate:

$$\sigma_T \circ \mathrm{suc} = (\sigma_{d,2}\sigma_{d,1})^{\frac{d-1}{2}} \circ {\sigma_{d,2}}^{-1}\sigma_{d,1} = (\sigma_{d,2}\sigma_{d,1})^{\frac{d+1}{2}} = \mathrm{suc} \circ \sigma_T$$

□

The two claims finish the proof of the fact that $T_n \in \Gamma_n$.

Step 4: $-I \in \Gamma_n$

We need to construct an affine map ϕ_{-I} that is locally a rotation by π. Let ϕ_{-I} denote the map that rotates every copy of X_n around the centre of its edge x_{n-1} by π. Clearly the map sends every copy of X_n to itself. The preimage of an edge x_i is send to x'_i. It remains to show, that this is consistent with the gluing of the copies according to the monodromy map. The monodromy

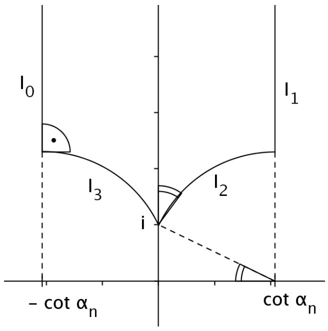
$m_{n,d}(x_i)$ for $i \notin \{k_1, k_2\}$ is trivial, so there is nothing to prove. The permutations $m_{n,d}(x_{k_1})$ and $m_{n,d}(x_{k_2})$ are involutions, so that the neighbour copy at x_i equals the neighbour copy at x'_i for all $i \in \{0, \ldots n-2\}$. It follows, that $-I \in \Gamma_n$.

Step 1 to 4 together with Lemma 4.2 and Lemma 2.1 prove Theorem 1 a).

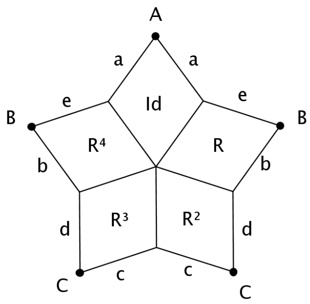
Knowing the Veech group of the translation surfaces $Y_{n,d}$ for all odd $n \geq 5$ and all $d \geq 2$, we can compute the basic properties of the Teichmüller curves belonging to the surfaces.

Corollary 4.3. $\mathbb{H}/\Gamma_n$ *has genus* 0 *and* $\frac{n+1}{2}$ *cusps.*

Proof. A fundamental region F of $\Gamma(X_n)$ is shown in Figure 12(a). Since $\Gamma(X_n) = \Gamma_n\, I \cup \Gamma_n\, R_n \cup \cdots \cup \Gamma_n\, {R_n}^{n-1}$, the fundamental region of Γ_n is $G = I(F) \cup R_n(F) \cup \cdots \cup {R_n}^{n-1}(F)$ (see e.g. [Kat92] Theorem 3.1.2). Figure 12(b) illustrates the fundamental region G of Γ_5 in a schematic picture. The elements T_5, $R_5T_5{R_5}^{-4}$, ${R_5}^2T_5{R_5}^{-3}$, ${R_5}^3T_5{R_5}^{-2}$ and ${R_5}^4T_5{R_5}^{-1}$ identify



(a) $\Gamma(X_n) = \langle T_n, R_n \rangle$



(b) Γ_5

Figure 12: Fundamental regions

the edges with labels a, b, c, d and e. The cusps in $\mathbb{R} \cup \infty$, i.e. the non equivalent vertices of the fundamental region of Γ_5, are labelled by the letters A to C, where $A = \{\infty\}$, $B = \{\cot \alpha_5, -\cot \alpha_5\}$ and $C = \{\cot 2\alpha_5, -\cot 2\alpha_5\}$, with $\alpha_n = \frac{\pi}{n}$. At the three cusps in $\mathbb{H}/\Gamma_5$, one, two and two copies of the fundamental region of $\Gamma(X_5)$ meet, respectively. We say the cusps have *relative width* 1, 2 and 2. In addition it can be seen, that the Euler characteristic $\chi(\mathbb{H}/\Gamma_5) = (3 + 4) - (2 \cdot 5) + 5 = 2$ and thus the genus of $\mathbb{H}/\Gamma_5$ is 0.

In general, the fundamental region G of Γ_n consists of n quadrilaterals with one cusp $A_0 = \{\infty\}$ of relative width 1 and $\frac{n-1}{2}$ cusps $A_i = \{\cot i\alpha_n, -\cot i\alpha_n\}$ for $i = 1, \ldots, \frac{n-1}{2}$ of relative width 2. In addition, $\mathbb{H}/\Gamma_n$ has a vertex at i and $\frac{n+1}{2}$ non equivalent vertices at $\cot \alpha_n + \frac{i}{\sin \alpha_n}$, $R(\cot \alpha_n + \frac{i}{\sin \alpha_n})$, $\ldots$, $R^{\frac{n-1}{2}}(\cot \alpha_n + \frac{i}{\sin \alpha_n})$. It follows, that $\chi(\mathbb{H}/\Gamma_n) = (\frac{n+1}{2} + 1 + \frac{n+1}{2}) - (2 \cdot n) + n = 2$ which implies $g(\mathbb{H}/\Gamma_n) = 0$. $\square$

4.3 The Veech group of the even series.

In this section, we explicitly determine the Veech group for $Y_{n,d}$ in the case that n is even. Just like in Section 4.2 the Veech group depends only on n and is generated by $-I$ and parabolic matrices. Hence we obtain Theorem 1 b) as analogue of Theorem 1 a).

Theorem 1 b). *For even $n \geq 8$, the Veech group of $Y_{n,d}$ is*

$$\begin{aligned}\Gamma_n := \;& \Gamma(Y_{n,d}) \\ = \;& \langle -I, T_n, {R_n}^2\, {T_n}^2\, {R_n}^{-2}, \ldots, {R_n}^{n-2}\, {T_n}^2\, {R_n}^{-(n-2)}, \, ({T_n}^{-1}{R_n}^2)^2, \\ & {R_n}^2\, ({T_n}^{-1}{R_n}^2)^2\, {R_n}^{-2}, \ldots, {R_n}^{n-2}\, ({T_n}^{-1}{R_n}^2)^2\, {R_n}^{-(n-2)} \rangle\ .\end{aligned}$$

The matrices R_n and T_n are defined as in Chapter 3.1 as

$$R_n = \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix} \text{ and } T_n = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix} \text{ where } \lambda_n = 2\cot\frac{\pi}{n}$$

and $I \in \mathrm{GL}_2(\mathbb{R})$ is the identity matrix.

The proof is very similar to the proof of Theorem 1 a). Hence we will state the outline of the proof while mainly referring to the corresponding proofs in Section 4.2.

Concept of proof: Using similar arguments as in Lemma 4.2 one obtains that Γ_n is an index $\frac{n}{2}$ subgroup of $\Gamma(X_n) = \langle {R_n}^2, T_n \rangle$ with coset representatives $I, {R_n}^2, {R_n}^4, \ldots, {R_n}^{n-2}$. We now have to repeat the steps 1 to 4. Step 3 and 4 work the same way as in Section 4.2. Thus it is sufficient to carry out step 1 and step 2.

Step 1: We have to find affine maps with parabolic derivatives ${R_n}^{2l}\, {T_n}^2\, {R_n}^{-2l}$ for $l \in \{1, \ldots, \frac{n-2}{2}\}$ and ${R_n}^{2l}\, ({T_n}^{-1}{R_n}^2)^2\, {R_n}^{-2l}$ for $l \in \{0, \ldots, \frac{n-2}{2}\}$. Let $v_j = {R_n}^j \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The matrix ${R_n}^{2l}\, {T_n}^2\, {R_n}^{-2l}$ is the derivative of the shearing in direction v_{2l} with factor $2\lambda_n$. A consequence of the relation $R_n {T_n}^{-1} R_n = -T_n$ is $({T_n}^{-1}{R_n}^2)^2 = {R_n}^{-1}{T_n}^2 R_n$, so the matrix ${R_n}^{2l}\, ({T_n}^{-1}{R_n}^2)^2\, {R_n}^{-2l}$ is the derivative of the shearing in direction v_{2l-1} with factor $2\lambda_n$. Hence we investigate the cylinder decompositions of X_n and $Y_{n,d}$ in the directions v_j for $j \in \{-1, 1, 2, \ldots, n-2\}$. In direction v_{2l-1}, X_n decomposes into $\frac{n}{4}$ cylinders if $n \equiv 0 \bmod 4$ and into $\frac{n+2}{4}$ cylinders, if $n \equiv 2 \bmod 4$. A short calculation shows that the innermost cylinder has inverse modulus $\frac{1}{2}\lambda_n$ and that all other cylinders have inverse modulus λ_n. Recall that the cylinders in the cylinder decomposition in direction v_{2l} have inverse modulus λ_n (see equation 3 on page 8). Only the vertical decomposition ($j = \frac{n}{2}$) contains a cylinder containing both x_{k_1} and x_{k_2} ($j \neq 0$ excludes the horizontal direction). We conclude that the arguments in the odd case also hold in the even case for the parabolic matrices ${R_n}^{2l}\, {T_n}^2\, {R_n}^{-2l}$ where $l \in \{1, \ldots, \frac{n-2}{2}\}$, $l \neq \frac{n}{4}$ and for ${R_n}^{2l}\, ({T_n}^{-1}{R_n}^2)^2\, {R_n}^{-2l}$ where $l \in \{0, \ldots, \frac{n-2}{2}\}$, $l \neq \frac{n+2}{4}$.

For $n \equiv 0 \bmod 4$ and $l = \frac{n}{4}$, the innermost cylinder in the decomposition of X_n in direction v_{2l} contains x_{k_1} and x_{k_2}. This cylinder has 1 or 2 preimage cylinders in $Y_{n,d}$ with inverse modulus $d\lambda_n$ or $\frac{d}{2}\lambda_n$, respectively (depending on d being

odd or even). If we rotate $Y_{n,d}$ and X_n clockwise by 90 degrees and introduce a new labelling of the preimages of X_n in $Y_{n,d}$ (see Figure 13), we can repeat the arguments of step 3 in Section 4.2 and find an affine map for $M = {R_n}^{\frac{n}{2}}\, T_n\, {R_n}^{-\frac{n}{2}}$. Then of course $M^2 = {R_n}^{\frac{n}{2}}\, {T_n}^2\, {R_n}^{-\frac{n}{2}}$ lies in $\Gamma(Y_{n,d})$. To find an affine map for

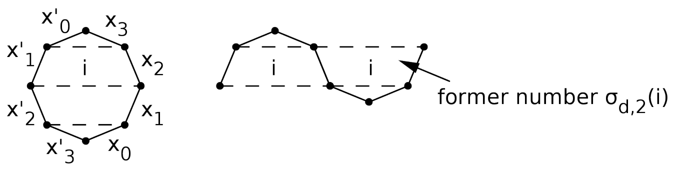


Figure 13: New labelling of the cylinders for X_8

M we have to find a permutation σ_M of the copies of X_n in $Y_{n,d}$ that fulfils the two conditions $\sigma_M(\mathrm{suc}(i)) = \mathrm{suc}(\sigma_M(i))$ and $\sigma_{d,2}(\sigma_M(i)) = \sigma_M(\sigma_{d,2}(\mathrm{pred}(i)))$ with $\mathrm{suc}(i) = \sigma_{d,1}(\sigma_{d,2}(i))$ and $\mathrm{pred} = \mathrm{suc}^{-1}$. The permutation σ_T^{-1} with σ_T as in step 3 in Section 4.2 fulfils these two conditions.

The same arguments prove that $M^2 = {R_n}^{\frac{n+2}{2}}\, ({T_n}^{-1}{R_n}^2)^2\, {R_n}^{-\frac{n+2}{2}}$ is contained in the Veech group of $Y_{n,d}$ if $n \equiv 2 \bmod 4$ ($l = \frac{n+2}{4}$).

Step 2: In the same way as in Section 4.2 we obtain that ${R_n}^{2l} \notin \Gamma(Y_{n,d})$ for $l \in \{1, \ldots, \frac{n-2}{2}\}$, $l \neq \frac{n}{4}$ if $d \neq 4$ or if $d = 4$ and $v_l = {R_n}^{2l} \cdot \begin{pmatrix}1\\0\end{pmatrix}$ is not parallel to x_{k_1} or x_{k_2}.

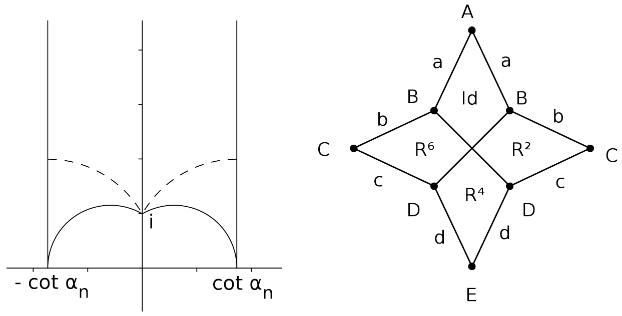
If $d = 4$ and v_l is parallel to x_{k_1} (or x_{k_2} respectively), then $n \equiv 2 \bmod 4$ and in particular $n \geq 10$. In this case one cylinder c in direction v_l contains the gluing along x_{k_2} (or x_{k_1} respectively). This cylinder has two preimage cylinders, both with inverse modulus $2\lambda_n$ and cylinder number $\frac{n-2}{4} - 1$ i.e. it has height $h = 2\cos(\frac{n-8}{2} \cdot \frac{\pi}{n})\sin(\frac{\pi}{n})$ (see equation 3 on page 8). The two cylinders in the horizontal decomposition of $Y_{n,4}$ with inverse modulus $2\lambda_n$ are preimages of cylinder $\frac{n-2}{4}$, so they have height $\tilde{h} = 2\cos(\frac{n-4}{2} \cdot \frac{\pi}{n})\sin(\frac{\pi}{n})$. Since $\tilde{h} - h = 2\sin(\frac{\pi}{n})(\cos(\frac{n-4}{2} \cdot \frac{\pi}{n}) - \cos(\frac{n-8}{2} \cdot \frac{\pi}{n})) = -4\sin(\frac{n-6}{2} \cdot \frac{\pi}{n})\sin^2(\frac{\pi}{n}) \neq 0$, no affine map with derivative ${R_n}^{2l}$ exists on $Y_{n,4}$.

If $n \equiv 0 \bmod 4$ and $l = \frac{n}{4}$, then v_l is vertical and the cylinder decomposition of $Y_{n,d}$ in direction v_l has a cylinder with height $h = 2\cos(\frac{\pi}{n})\sin(\frac{\pi}{n})$ and inverse modulus $d\lambda_n$ or two cylinders with inverse modulus $\frac{d}{2}\lambda_n$, respectively (depending on d being odd or even). The cylinder with inverse modulus $d\lambda_n$ or $\frac{d}{2}\lambda_n$ in the horizontal decomposition of $Y_{n,d}$ has height $\tilde{h} = 2\cos(\frac{n-2}{2} \cdot \frac{\pi}{n})\sin(\frac{\pi}{n})$ which is different from h. Hence we may conclude that no affine map with derivative ${R_n}^{\frac{n}{2}}$ exists on $Y_{n,d}$. □

We deduce the following Corollary on the Teichmüller curve to $Y_{n,d}$.

Corollary 4.4. *$\mathbb{H}/\Gamma_n$ has genus 0 and $\frac{n+2}{2}$ cusps.*

Proof. The Veech group $\Gamma(X_n) = \langle T_n, {R_n}^2 \rangle$ has index two in the Hecke group $\langle T_n, R_n \rangle$. In particular it has a fundamental region F as shown in Figure 14(a) with 2 cusps, $\{\infty\}$ and $\{-\cot\alpha_n, \cot\alpha_n\}$ with $\alpha_n = \frac{\pi}{n}$. A fundamental region of Γ_n is $\widetilde{F} = F \cup {R_n}^2(F) \cup \cdots \cup {R_n}^{n-2}(F)$. The vertices of $\widetilde{F}$ in $\mathbb{R} \cup \{\infty\}$



(a) $\Gamma(X_n) = \langle T_n, {R_n}^2 \rangle$ (b) Γ_8

Figure 14: Fundamental regions

are $\{\infty\}$ and $\cot(r\,\alpha_n)$ with $r \in \{1, \ldots, n-1\}$. The matrices ${R_n}^{2l}\,{T_n}^2\,{R_n}^{-2l}$ and ${R_n}^{2l}\,({T_n}^{-1}{R_n}^2)^2\,{R_n}^{-2l}$ identify $\frac{n-2}{2}$ pairs of vertices (see Figure 14(b) for Γ_8), so $\mathbb{H}/\Gamma_n$ has $\frac{n+2}{2}$ cusps. Using the Euler characteristic, it can be seen that $\mathbb{H}/\Gamma_n$ has genus 0. $\square$

5 Infinite translation surfaces.

In this section we study the limit of our covering families and show that their Veech groups are lattices in $\mathrm{SL}_2(\mathbb{R})$. An *infinite translation surface* X^∞ can be obtained by gluing countably infinitely many polygons via identification of edge pairs using translations. We only consider the case where countably infinitely many copies of one polygon are glued together. Then, if infinitely many copies are glued at one vertex, an infinite angle singularity arises. Let Σ^∞ be the set of infinite angle singularities, then $X^\infty \setminus \Sigma^\infty$ is a Riemann surface and X^∞ is the metric completion of $X^\infty \setminus \Sigma^\infty$. So we get a so called *tame flat surface.* For more details on infinite translation surfaces see e.g. [Val09].

As in the finite case, let $n \geq 5$, $n \neq 6$. To simplify notation, let $X^* = X \setminus \Sigma(X)$ for a finite or tame infinite translation surface X. Our series of translation coverings give rise to the infinite translation surface $Y_{n,\infty}$, defined by the monodromy

$$m_{n,\infty} : \left\{ \begin{array}{rcl} \pi_1(X_n^*) & \longrightarrow & \mathrm{Sym}(\mathbb{Z}) \\ x_i & \mapsto & \mathrm{id} \quad, \text{for } i \notin \{k_1, k_2\} \\ x_{k_1} & \mapsto & \sigma_{k_1} \\ x_{k_2} & \mapsto & \sigma_{k_2} \end{array} \right.$$

where $\sigma_{k_1} : l \mapsto \left\{ \begin{array}{ll} l+1 & , l \text{ even} \\ l-1 & , l \text{ odd} \end{array} \right.$, $\sigma_{k_2} : l \mapsto \left\{ \begin{array}{ll} l-1 & , l \text{ even} \\ l+1 & , l \text{ odd} \end{array} \right.$,

$$k_1 = \left\{ \begin{array}{ll} \frac{n-1}{2} & , n \text{ odd} \\ \frac{n}{4} - 1 & , n \equiv 0 \bmod 4 \\ \frac{n-2}{4} - 1 & , n \equiv 2 \bmod 4 \end{array} \right. \quad \text{and} \quad k_2 = \left\{ \begin{array}{ll} \frac{n+1}{2} & , n \text{ odd} \\ \frac{n}{4} & , n \equiv 0 \bmod 4 \\ \frac{n-2}{4} + 1 & , n \equiv 2 \bmod 4 \end{array} \right. .$$

Remark 5.1. The surface $Y_{n,\infty}$ has 4 infinite angle singularities.

Proof. For odd n and even n, $n \equiv 0 \bmod 4$, a simple clockwise path around the singularity in X_n is given by

$$p = x_0 {x_1}^{-1} x_2 {x_3}^{-1} x_4 \dots x_{n-3} {x_{n-2}}^{-1} {x_0}^{-1} x_1 {x_2}^{-1} x_3 \dots {x_{n-3}}^{-1} x_{n-2} \qquad \text{or}$$
$$p = x_0 {x_1}^{-1} x_2 {x_3}^{-1} x_4 \dots x_{\frac{n}{2}-2} {x_{\frac{n}{2}-1}}^{-1} {x_0}^{-1} x_1 {x_2}^{-1} x_3 \dots {x_{\frac{n}{2}-2}}^{-1} x_{\frac{n}{2}-1} \qquad ,$$

respectively. Since the monodromy of x_i is the identity iff $i \notin \{k_1, k_2\}$, $\sigma_{k_1} = {\sigma_{k_1}}^{-1}$ and $\sigma_{k_2} = {\sigma_{k_2}}^{-1}$, we have

$$(m_{n,\infty}(p))(l) = (\sigma_{k_2} \circ \sigma_{k_1})^2(l) = \begin{cases} l+4 & , l \text{ even} \\ l-4 & , l \text{ odd} \end{cases} .$$

For even n, $n \equiv 2 \bmod 4$, two simple clockwise paths p_1 and p_2 around the two singularities in X_n are given by $p_1 = x_0 {x_1}^{-1} x_2 {x_3}^{-1} x_4 \dots {x_{\frac{n}{2}-2}}^{-1} x_{\frac{n}{2}-1}$ and $p_2 = x_1 {x_2}^{-1} x_3 \dots x_{\frac{n}{2}-2} {x_{\frac{n}{2}-1}}^{-1} {x_0}^{-1}$. It follows

$$(m_{n,\infty}(p_1))(l) = (m_{n,\infty}(p_2))(l) = \sigma_{k_2}(\sigma_{k_1}(l)) = \begin{cases} l+2 & , l \text{ even} \\ l-2 & , l \text{ odd} \end{cases} .$$

□

Remark 5.2. The covering $p_{n,\infty} : Y_{n,\infty} \to X_n$ is not a $\mathbb{Z}$-covering, but $Y_{n,\infty}$ is a $\mathbb{Z}$-covering of $Y_{n,2}$: A basis of $\pi_1(Y^*_{n,2})$ is $B = \{x_{k_2} {x_{k_1}}^{-1}, {x_{k_1}}^2, x_{k_1} x_{k_2}\} \cup \{x_i,\ x_{k_1} x_i {x_{k_1}}^{-1} \mid i \notin \{k_1, k_2\}\}$ and $\pi_1(Y^*_{n,\infty}) \subseteq \pi_1(Y^*_{n,2})$. Every copy of $Y_{n,2}$ in $Y_{n,\infty}$ consists of two copies of X_n with numbers $2l$ and $2l+1$. Since $\sigma_{k_2}\sigma_{k_1}(2l) = 2l+2$ and ${\sigma_{k_1}}^{-1}\sigma_{k_2} = (\sigma_{k_2}\sigma_{k_1})^{-1}$, the image of $\pi_1(Y^*_{n,2})$ under $m_{n,\infty}$ restricted to the even numbers defines a transitive permutation group on $2\mathbb{Z}$ isomorphic to $\mathbb{Z}$. It follows that

$$\tilde{m} = m_{n,\infty}|_{\pi_1(Y^*_{n,2})} = \begin{cases} \pi_1(Y^*_{n,2}) & \to & \mathrm{Sym}(2\mathbb{Z}) \cong \mathrm{Sym}(\mathbb{Z}) \\ x_{k_1} x_{k_2} & \mapsto & (2l \mapsto 2l+2) \\ x_{k_2} {x_{k_1}}^{-1} & \mapsto & (2l \mapsto 2l-2) \\ w & \mapsto & \mathrm{id} \qquad , w \in B \setminus \{x_{k_1} x_{k_2}, x_{k_2} {x_{k_1}}^{-1}\} \end{cases}$$

defines a normal covering $\tilde{p} : Y_{n,\infty} \to Y_{n,2}$ with Galois group $\mathbb{Z}$ and $p_{n,\infty} = p_{n,2} \circ \tilde{p}$.

In [HW09] $\mathbb{Z}$-covers are defined by non-zero elements in the relative homology $H_1(X, \Sigma(X); \mathbb{Z})$. In our case, a representative for the element $w \in H_1(Y_{n,2}, \Sigma(Y_{n,2}); \mathbb{Z})$, defining $Y_{n,\infty}$, consists of two reverse paths on the edges x_{k_2} in copy 0 and copy 1. For $n = 8$ this can be seen in Figure 15. The holon-

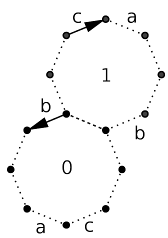


Figure 15: $w \in H_1(Y_{8,2}, \Sigma(Y_{8,2}); \mathbb{Z})$, defining $Y_{8,\infty}$

omy $hol(w) = 0$, so according to [HW09] the cover is recurrent, i.e. the straight line flow on $Y_{n,\infty}$ is recurrent for almost every direction θ.

We want to determine the Veech group of $Y_{n,\infty}$ and start by arguing, why $\Gamma(Y_{n,d})$ is contained in $\Gamma(X_n)$, even for $d = \infty$. For this we use the developments of the saddle connections of a translation surface (X, ω) (see [Vor96] Chapter 3.1). For every saddle connection there is a chart containing the saddle connection without end points. Thus the saddle connection defines an interval in $\mathbb{R}^2$, i.e. a vector and its additive inverse in $\mathbb{R}^2$, not depending on the chosen chart. The set of all theses vectors is called $SC(X)$.

Lemma 5.3. *For $n \geq 8$, the Veech group $\Gamma(Y_{n,d})$ of $Y_{n,d}$ is contained in $\Gamma(X_n)$, even for $d = \infty$.*

Proof. The covering map $p_{n,d} : Y_{n,d} \to X_n$ is locally a translation, so every saddle connection in $Y_{n,d}$ is mapped one to one to a saddle connection in X_n and every saddle connection in X_n can be lifted to a saddle connection in $Y_{n,d}$. It immediately follows $SC(X_n) = SC(Y_{n,d})$. This idea, at least for finite d, and the following statement can be found in [Vor96]. Affine maps preserve the set of saddle connections. We conclude that $\Gamma(Y_{n,d})$ is contained in the stabiliser $\mathrm{Stab}(SC(Y_{n,d}))$ of $SC(Y_{n,d})$ and $\Gamma(X_n) \subseteq \mathrm{Stab}(SC(X_n))$, respectively. According to [Vor96] Proposition 3.1, $SC(X_n)$ has no limit points, hence $\mathrm{Stab}(SC(X_n))$ is a discrete group.

For odd n, the Veech group $\Gamma(X_n)$ is a Hecke group. In particular it is a maximal discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ and $\Gamma(X_n) \subset \mathrm{Stab}(SC(X_n))$ implies $\Gamma(X_n) = \mathrm{Stab}(SC(X_n))$ which proves the statement for n odd.

For even n, $\Gamma(X_n)$ is a triangle group with signature $(\frac{n}{2}, \infty, \infty)$. According to [Sin72] Theorem 1 and Theorem 2, the only possible discrete groups properly containing $\Gamma(X_n)$ are the triangle groups with signature $(n, 2, \infty)$. Let G denote the triangle group $(n, 2, \infty)$, containing $\Gamma(X_n)$. Then $G = \langle T_n, R_n \rangle$ and $[G : \Gamma(X_n)] = 2$. The shortest saddle connections in X_n are the edges of the regular n-gon. Their corresponding elements in $SC(X_n)$ are

$$\begin{aligned} v_l &= {R_n}^{2(l+1)} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - {R_n}^{2l} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(2(l+1)\frac{\pi}{n}) - \cos(2l\frac{\pi}{n}) \\ \sin(2(l+1)\frac{\pi}{n}) - \sin(2l\frac{\pi}{n}) \end{pmatrix} \\ &= \begin{pmatrix} -2\sin((2l+1)\frac{\pi}{n})\sin(\frac{\pi}{n}) \\ 2\cos((2l+1)\frac{\pi}{n})\sin(\frac{\pi}{n}) \end{pmatrix}. \end{aligned}$$

The rotation R_n doesn't change the length of a saddle connection, but

$$R_n \cdot v_l = {R_n}^{2(l+1)+1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - {R_n}^{2l+1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2\sin((2l+2)\frac{\pi}{n})\sin(\frac{\pi}{n}) \\ 2\cos((2l+2)\frac{\pi}{n})\sin(\frac{\pi}{n}) \end{pmatrix} \neq v_j$$

for all $j \in \{0, \ldots, n-1\}$. So $R_n \notin \mathrm{Stab}(SC(X_n))$ and $\mathrm{Stab}(SC(X_n)) = \Gamma(X_n)$. We conclude $\Gamma(Y_{n,d}) \subset \mathrm{Stab}(SC(Y_{n,d})) = \mathrm{Stab}(SC(X_n)) = \Gamma(X_n)$. □

Now we can calculate the Veech group of the infinite covering surface by reproducing the arguments of the finite case.

Theorem 2. *For $n \geq 5$, $n \neq 6$, the Veech group $\Gamma(Y_{n,\infty})$ of $Y_{n,\infty}$ is Γ_n. In particular $Y_{n,\infty}$ is an infinite translation surface with a lattice Veech group.*

Proof. We reconsider the steps 1 to 4 of the proof in the finite case. In step 1 we can use identical arguments to show that ${R_n}^l\, {T_n}^2\, {R_n}^{-l}$ (if n is odd) or that ${R_n}^{2l}\, {T_n}^2\, {R_n}^{-2l}$ and ${R_n}^{2l}\, ({T_n}^{-1}{R_n}^2)^2\, {R_n}^{-2l}$ (if n is even) are contained

in $\Gamma(Y_{n,\infty})$. Once again the proof for ${R_n}^{\frac{n}{2}}\,{T_n}^2\,{R_n}^{-\frac{n}{2}}$ if $n \equiv 0 \bmod 4$ and for ${R_n}^{\frac{n+2}{2}}\,({T_n}^{-1}{R_n}^2)^2\,{R_n}^{-\frac{n+2}{2}}$ if $n \equiv 2 \bmod 4$ rely on σ_T (as defined below).

To prove that $I, R_n, \ldots, {R_n}^{n-1}$ (or $I, {R_n}^2, \ldots, {R_n}^{n-2}$ respectively) lie in different cosets of $\Gamma(Y_{n,\infty})\backslash\Gamma(X_n)$ in step 2, we compute the monodromy of the core curve p of cylinder

$$k = \begin{cases} \frac{n-1}{2} & , n \text{ odd} \\ \frac{n}{4} & , n \equiv 0 \bmod 4 \\ \frac{n-2}{4} & , n \equiv 2 \bmod 4 \end{cases}$$

of the horizontal cylinder decomposition of X_n:

$$m_{n,\infty}(p) = m_{n,\infty}(x_{k_1}{x_{k_2}}^{-1}) = {\sigma_{k_2}}^{-1} \circ \sigma_{k_1} = \begin{cases} i \mapsto i+2 & , i \text{ even} \\ i \mapsto i-2 & , i \text{ odd} \end{cases}$$

So cylinder k has two preimages of infinite length in $Y_{n,\infty}$. Recall that no cylinder of X_n in the decomposition in direction v_l contains both x_{k_1} and x_{k_2} if n is odd or if n is even and $l \neq \frac{n}{4}$. This implies that $Y_{n,\infty}$ contains no infinite cylinder (i.e. no strips) in these directions. As in the finite case $n \equiv 0 \bmod 4$ and $l = \frac{n}{4}$ is an exception. There the cylinder decomposition in direction v_l contains two infinite cylinders. But their height differs from the height of the infinite cylinders in the horizontal decomposition. We conclude that there is no affine map with derivative ${R_n}^l$ (or ${R_n}^{2l}$) on $Y_{n,\infty}$.

The proof of $T_n \in \Gamma(Y_{n,\infty})$ is similar to the finite case when d is even. We consider the two infinite length preimages of cylinder k of the horizontal cylinder decomposition. The maps suc and pred are defined as in the finite case and again we need to prove the existence of a permutation $\sigma_T \in \mathrm{Sym}(\mathbb{Z})$ fulfilling the two properties: $\sigma_T(\mathrm{suc}(i)) = \mathrm{suc}(\sigma_T(i))$ and $m_{n,\infty}(x_{k_1})(\sigma_T(i)) = \sigma_T(m_{n,\infty}(x_{k_2})(i))$. Let

$$\sigma_T = \begin{cases} i \mapsto i & , i \text{ even} \\ i \mapsto i+2 & , i \text{ odd} \end{cases} .$$

The map $\mathrm{suc} = m_{n,\infty}(x_{k_1}{x_{k_2}}^{-1})$ was calculated above. It remains to check the two conditions on σ_T:

$$\sigma_T(\mathrm{suc}(i)) = \begin{cases} i+2 & , i \text{ even} \\ i & , i \text{ odd} \end{cases} = \mathrm{suc}(\sigma_T(i))$$

$$m_{n,\infty}(x_{k_1})(\sigma_T(i)) = i+1 = \sigma_T(m_{n,\infty}(x_{k_2})(i))$$

Finally the proof for $-I \in \Gamma(Y_{n,\infty})$ in step 4 works as in the finite case. Lemma 5.3 and Lemma 4.2 complete the proof. □

References

[Fre08] Myriam Freidinger, *Stabilisatorgruppen in Aut(F_z) und Veechgruppen von Überlagerungen*, diploma thesis, Universität Karlsruhe, 2008.

[GJ00] E. Gutkin and C. Judge, *Affine mappings of translation surfaces: Geometry and arithmetic*, Duke Math. J. **103** (2000), no. 2, 191–213.

[Hoo08] P. Hooper, *Dynamics on an infinite surface with the lattice property*, 2008, arXiv:0802.0189v1.

[HS01] P. Hubert and T.A. Schmidt, *Invariants of translation surfaces*, Annales de l'institut Fourier **51** (2001), no. 2, 461–495.

[HS07] F. Herrlich and G. Schmithüsen, *On the boundary of Teichmüller disks in Teichmüller and in Schottky space*, Handbook of Teichmüller Theory **I** (2007), 293–349.

[HS10] P. Hubert and G. Schmithüsen, *Infinite translation surfaces with infinitely generated Veech groups*, J. Mod. Dyn. **4** (2010), no. 4, 715–732.

[HW09] P. Hooper and B. Weiss, *Generalized staircases: recurrence and symmetry*, 2009, arXiv:0905.3736v1.

[Joh97] D.L. Johnson, *Presentations of groups*, second ed., Cambridge University Press, 1997.

[Kat92] S. Katok, *Fuchsian Groups*, University of Chicago Press, 1992.

[MÖ06] M. Möller, *Periodic points on Veech surfaces and the Mordell-Weil group over a Teichmüller curve*, Invent. Math. **165** (2006), no. 3, 633–649.

[McM06] C.T. McMullen, *Prym varieties and Teichmüller curves*, Duke Math. J. **133** (2006), no. 3, 569–590.

[Mir95] R. Miranda, *Algebraic Curves and Riemann Surfaces*, American Mathematical Society, 1995.

[Sch04] G. Schmithüsen, *An algorithm for finding the Veech group of an origami*, Experimental Mathematics **13** (2004), no. 4, 459–472.

[Sch06] ———, *Examples for Veech groups of origamis*, Proceedings of the III Iberoamerican Congress on Geometry. In: The Geometry of Riemann Surfaces and Abelian Varieties. Contemp. Math. **397** (2006), 193–206.

[Sin72] D. Singerman, *Finitely maximal Fuchsian groups*, J. London Mathematical Society **s2-6** (1972), 29–38.

[Val09] F. Valdez, *Veech groups, irrational billiards and stable abelian differentials*, 2009, arXiv:0905.1591v2.

[Vee89] W.A. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. math. **97** (1989), 553–583.

[Vor96] Y.B. Vorobets, *Planar structures and billiards in rational polygons: the Veech alternative*, Russ. Math. Surv. **51** (1996), 779–817.

[ZK75] A.N. Zemlyakov and A.B. Katok, *Topological transitivity of billiards in polygons*, Math. Notes **18** (1975), 760–764.